SMOOTH PARTITIONS OF ANOSOV DIFFEOMORPHISMS ARE WEAK BERNOULLI

BY
RUFUS BOWEN†

ABSTRACT

It is shown that smooth partitions are weak Bernoulli for C^2 measure preserving Anosov diffeomorphisms. A related type of coding is defined and an invariant discussed.

We prove here the conjecture of Benjamin Weiss [9] that smooth partitions of Anosov diffeomorphisms are weak Bernoulli. A strong type of coding will be used for this purpose.

Let T be an automorphism of the probability space (X, m). For $\mathcal{P}$ and $\mathcal{Q}$ (finite) partitions of X one writes $\mathcal{P} \perp_\delta \mathcal{Q}$ if

$$\sum_{P \in \mathcal{P}, Q \in \mathcal{Q}} |m(P \cap Q) - m(P)m(Q)| < \delta.$$

A partition $\mathcal{A}$ is *weak Bernoulli* for T if for every $\varepsilon > 0$ there is an $L = L(\varepsilon)$ so that

$$\bigvee_{j=-n}^{0} T^{-j}\mathcal{A} \perp_\varepsilon \bigvee_{j=L}^{L+n} T^{-j}\mathcal{A} \quad \text{for all} \quad n \geqq 0.$$

This property originated as a way to prove Bernoulliness [4] and is now seen as a strong statement about certain partitions for Bernoulli shifts [11].

LEMMA 1. *If $\mathcal{P} \perp_\delta \mathcal{Q}$ and $\mathcal{R} \subset_\varepsilon \mathcal{P}$, then $\mathcal{R} \perp_{\delta+2\varepsilon} \mathcal{Q}$.*

PROOF. Let $\mathcal{R} = \{R_1, \cdots, R_k\}$, $\mathcal{R}' = \{R'_1, \cdots, R'_k\}$, where $\mathcal{R}' \subset \mathcal{P}$ and $\Sigma_i m(R_i \Delta R'_i) < \varepsilon$.

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$$|m(R_i \cap Q) - m(R_i)m(Q)|$$
$$\leqq |m(R_i \cap Q) - m(R'_i \cap Q)| + |m(R'_i \cap Q) - m(R'_i)m(Q)|$$
$$+ |m(R'_i)m(Q) - m(R_i)m(Q)|$$
$$\leqq m((R_i \Delta R'_i) \cap Q) + |m(R'_i \cap Q) - m(R'_i)m(Q)| + m(Q)m(R_i \Delta R'_i).$$

Now $\mathscr{R}' \perp_\delta \mathscr{Q}$ because $\mathscr{R}' \subset \mathscr{P}$. Summing the above inequality over i and $Q \in \mathscr{Q}$, we get

$$\sum_{i,Q} |m(R_i \cap Q) - m(R_i)m(Q)| \leqq \varepsilon + \delta + \varepsilon.$$

LEMMA 2. *If $\mathscr{P} \perp_\delta \mathscr{Q}$, $\mathscr{R} \subset_\varepsilon \mathscr{P}$ and $\mathscr{S} \subset_\varepsilon \mathscr{Q}$, then $\mathscr{R} \perp_{\delta+4\varepsilon} \mathscr{S}$.*

PROOF. By Lemma 1 one has $\mathscr{R} \perp_{\delta+2\varepsilon} \mathscr{Q}$. Then, applying Lemma 1 to $\mathscr{S} \subset_\varepsilon \mathscr{Q}$, one gets $\mathscr{R} \perp_{\delta+4\varepsilon} \mathscr{S}$.

DEFINITION. Let $\mathscr{A}$, $\mathscr{B}$ be two partitions of X. We say $\mathscr{A}$ is *boundedly coded* by $\mathscr{B}$ w.r.t. T if, for every $\varepsilon > 0$, there is a $K = K(\varepsilon)$ with

$$\bigvee_{j=0}^{n} T^{-j}\mathscr{A} \subset_\varepsilon \bigvee_{j=-K}^{n+K} T^{-j}\mathscr{B} \quad \text{for all } n.$$

PROPOSITION. 3. *If $\mathscr{A}$ is boundedly coded by $\mathscr{B}$ and $\mathscr{B}$ is weak Bernoulli, then $\mathscr{A}$ is also weak Bernoulli.*

PROOF. Fix $\varepsilon > 0$ and choose K as above. Let L be large. We have

$$\bigvee_{j=-n}^{0} T^{-j}\mathscr{A} \subset_\varepsilon \bigvee_{j=-n-K}^{K} T^{-j}\mathscr{B} \quad \text{and} \quad \bigvee_{j=L}^{L+n} T^{-j}\mathscr{A} \subset_\varepsilon \bigvee_{j=L-K}^{L+n+K} T^{-j}\mathscr{B}.$$

Since $\mathscr{B}$ is weak Bernoulli, for large L one has

$$\bigvee_{j=-n-K}^{K} T^{-j}\mathscr{B} \perp_\varepsilon \bigvee_{j=L-K}^{L+n+K} T^{-j}\mathscr{B}.$$

By Lemma 2 one then has

$$\bigvee_{j=-n}^{0} T^{-j}\mathscr{A} \perp_{5\varepsilon} \bigvee_{j=L}^{L+n} T^{-j}\mathscr{A} \quad \text{for all} \quad n \geqq 0,$$

i.e. $\mathscr{A}$ is weak Bernoulli.

We will call a finite partition of a manifold M *smooth* if the boundary of each set in the partition is a compact piecewise smooth differentiable submanifold.

THEOREM 4. *Suppose $T: M \to M$ is a C^2 Anosov diffeomorphism preserving a smooth invariant measure m. Then every smooth partition of M is weak Bernoulli for T.*

PROOF. Let $\mathcal{A}$ be the partition and $\partial_\varepsilon \mathcal{A}$ be the set of points within ε of the boundary of some member of $\mathcal{A}$. The smoothness condition gives us a constant c so that

$$m(\partial_\varepsilon \mathcal{A}) \leqq c\varepsilon.$$

Because T is Anosov (see [2]), there are positive constants α, b, λ with $\lambda \in (0,1)$ so that

$$\text{diam } E \leqq b\lambda^r \text{ whenever } E \subset M \text{ satisfies}$$

$$\text{diam } T^j E \leqq \alpha \text{ for all } j \in [-r, r].$$

Let $\mathcal{B}$ be any partition of M whose sets have diameter less than α.

Consider a set $E \in \bigvee_{i=-K}^{n+K} T^{-i}\mathcal{B}$. For $k \in [0, n]$ one has diam $T^j(T^k E) \leqq \alpha$ for all $|j| \leqq K + \min\{k, n-k\}$; hence diam $T^k E \leqq b\lambda^{K+\min\{k,n-k\}}$. Either $T^k E$ is totally contained within a single member of $\mathcal{A}$, or $T^k E \subset \partial_{b\lambda^{K+\min\{k,n-k\}}}\mathcal{A}$. The totality of all $E \in \bigvee_{i=-K}^{n+K} T^{-i}\mathcal{B}$ for which $T^k E$ hits more than one member of $\mathcal{A}$ have total measure at most $cb\lambda^{K+\min\{k,n-k\}}$; the totality of all E's for which this happens for *some* $k \in [0, n]$ have measure at most

$$\sum_{k=0}^{n} cb\lambda^{K+\min\{k,n-k\}} \leqq 2cb \sum_{j=0}^{\infty} \lambda^{K+j} = a\lambda^K,$$

where $a = 2cb(1-\lambda)^{-1}$. Any other E lies inside a single atom of $\bigvee_{k=0}^{n} T^{-k}\mathcal{A}$; hence one has

$$\bigvee_{k=0}^{n} T^{-k}\mathcal{A} \subset_{a\lambda^K} \bigvee_{i=-K}^{n+K} T^{-i}\mathcal{B}.$$

We have shown that $\mathcal{A}$ is boundedly coded by any small partition $\mathcal{B}$. The theory of Markov partitions [7] gives us partitions $\mathcal{B}$ of arbitrarily small diameter which are weak Bernoulli [3]. (Warning: "Markov" partitions are not always Markov in the probabilistic sense.) By Proposition 3, $\mathcal{A}$ is weak Bernoulli.

REMARK 5. For partitions $\mathcal{P}$, $\mathcal{Q}$ let

$$\beta(\mathcal{P}, \mathcal{Q}) = \sum_{P \in \mathcal{P}, Q \in \mathcal{Q}} |m(P \cap Q) - m(P)m(Q)|.$$

Set

$$\gamma_N(\mathcal{B}) = \sup_{n \geqq 0} \beta\left(\bigvee_{j=-n}^{0} T^{-j}\mathcal{B}, \bigvee_{j=N}^{n+N} T^{-j}\mathcal{B} \right).$$

In the above proof one gets that, for $K < N/2$,

$$\gamma_N(\mathscr{A}) \leqq 4a\lambda^K + \gamma_{N-2K}(\mathscr{B}).$$

Sometimes a weak Bernoulli partition $\mathscr{B}$ (i.e. $\gamma_N(\mathscr{B}) \to 0$ as $N \to \infty$) will satisfy the stronger condition

(*) There are positive constants u, v so that $\gamma_N(\mathscr{B}) \leqq ue^{-vN}$ for all $N \geqq 0$.

Taking $K = [N/4]$, one sees that $\mathscr{A}$ satisfies (*) if $\mathscr{B}$ does. The Markov partition partition $\mathscr{B}$ in fact does satisfy (*) [8], [6].

We will write $\mathscr{A} \sim_{\text{bd.}} \mathscr{B}$ if the two partitions $\mathscr{A}$ and $\mathscr{B}$ boundedly code each other. Then $\sim_{\text{bd.}}$ is an equivalence relation and one can look for invariants to distinguish $\sim_{\text{bd.}}$-equivalence classes of partitions. For any finite partition $\mathscr{P}$ one defines $I_{\mathscr{P}}: X \to R$ by

$$I_{\mathscr{P}}(x) = \sum_{P \in \mathscr{P}} (-\ln m(P))\chi_P.$$

Let

$$I_{\mathscr{A},n,T} = I_{\vee_{j=0}^{n-1} T^{-j}\mathscr{A}}$$

and define the distribution function $F_{\mathscr{A},n,T}$ on R by

$$F_{\mathscr{A},n,T}(r) = m\{x \in X: n^{-\frac{1}{2}}(I_{\mathscr{A},n,T}(x) - nh(T,\mathscr{A})) < r\}.$$

Proposition 6. *Suppose that $F_{\mathscr{A},n,T} \to F$ in distribution as $n \to \infty$. If $\mathscr{A} \sim_{\text{bd.}} \mathscr{B}$, then also $F_{\mathscr{B},n,T} \to F$ in distribution.*

Proof. If $\mathscr{A} \sim_{\text{bd.}} \mathscr{B}$, then $\mathscr{A}$ and $\mathscr{B}$ generate the same σ-algebras under T and $h(T,\mathscr{A}) = h(T,\mathscr{B})$. Given $\varepsilon > 0$, the fact

$$\bigvee_{j=0}^{n} T^{-j}\mathscr{A} \subset_{\varepsilon} \bigvee_{-K(\varepsilon)}^{n+K(\varepsilon)} T^{-j}\mathscr{B}$$

implies that $(1 - 1\sqrt{\varepsilon})\%$ of the atoms of $\vee_{-K(\varepsilon)}^{n+K(\varepsilon)} T^{-j}\mathscr{B}$ have at least $(1-\sqrt{\varepsilon})\%$ of their mass lying in a single atom of $\vee_{j=0}^{n} T^{-j}\mathscr{A}$. This yields

$$F_{\mathscr{A},n,T}(r) \leqq \sqrt{\varepsilon} + F_{\mathscr{B},n+2K(\varepsilon),T}(r - n^{-1/2}\ln(1-\sqrt{\varepsilon})).$$

If $L(\varepsilon)$ satifies

$$\bigvee_{j=0}^{n} T^{-j}\mathscr{B} \subset_{\varepsilon} \bigvee_{-L(\varepsilon)}^{n+L(\varepsilon)} T^{-j}\mathscr{A} \quad \text{for all } n \geqq 0,$$

one gets analogously

$$F_{\mathscr{B},n,T}(r) \leqq \sqrt{\varepsilon} + F_{\mathscr{A},n+2L(\varepsilon),T}(r - n^{-1/2}\ln(1-\sqrt{\varepsilon})).$$

The proposition follows from the two inequalities when $n \to \infty$.

For a Bernoulli partition $\mathcal{A}$ and $n \geqq 0$, let $X_n(x) = I_{\mathcal{A}}(T^n x)$. Then $X_0, X_1, \cdots$ is an independent sequence of random variables with the same distribution,

$$I_{\mathcal{A},n,T} = \sum_{j=0}^{n-1} X_j$$

and

$$EX_j = h(T, \mathcal{A}).$$

By the central limit theorem $F_{\mathcal{A},n,T}$ converges in distribution to $F_{\mathcal{A}}$ which is the centered normal law with the same variance as $I_{\mathcal{A}}$. By Proposition 6, Bernoulli partitions with probabilities $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and $(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ cannot boundedly code each other (in fact neither can boundedly code the other one by a slightly finer argument). On the other hand, the Meshalkin code [10] codes these partitions finitistically [9].

For another example we look at ergodic automorphisms of the 2-torus [1]. These are Anosov diffeomorphisms and here the Markov partitions are themselves smooth partitions. Adler and Weiss have constructed bounded codings between Markov partitions $\mathcal{B}$ for any two of these examples with the same entropy; hence any two small smooth partitions of two such examples boundedly code each other.

For Anosov automorphisms of the n-torus the situation is more complicated. Here the Markov partitions $\mathcal{B}$ [7] probably are not smooth, though they might still satisfy the condition $m(\partial_\varepsilon \mathcal{B}) \leqq c\varepsilon$ †. Thus it is unknown whether $\mathcal{B}$ is boundedly coded by small smooth partitions. It is also unknown here whether Markov partitions for different Anosov automorphisms with the same entropy boundedly code each other. Doug Lind suggested to us one thing that can be proven: an Anosov automorphism of the n-torus cannot have a smooth partition $\mathcal{A}$ which is a Bernoulli generator. If such an $\mathcal{A}$ existed, then $h(T, \mathcal{A}) = h(T, \mathcal{B}) = h(T)$ and, since $\mathcal{B}$ boundedly codes $\mathcal{A}$, one would have

$$\lim_n F_{\mathcal{A},n,T} \leqq \lim_n F_{\mathcal{B},n,T},$$

provided these distribution limits exist (see the proof of 8). There are constants k_1, k_2 so that ([7], [1, p. 13])

$$I_{\mathcal{B},n,T} - n \log \lambda \in [k_1, k_2],$$

† *Added in proof*: M. Ratner has shown that $m(\partial_\varepsilon \mathcal{B}) \leqq c\varepsilon$ for Anosov automorphisms of the n-torus.

where $h(T) = \log \lambda$. Hence $\lim_n F_{\mathcal{B},n,T}$ concentrates all mass at the origin. Because $\mathcal{A}$ is Bernoulli, $\lim_n F_{\mathcal{A},n,T}$ is normal or concentrates all mass at the origin. Since $\lim_n F_{\mathcal{A},n,T} \leqq \lim_n F_{\mathcal{B},n,T}$, $I_{\mathcal{A}}$ must in fact have variance 0; i.e., $\mathcal{A}$ has probabilities $(1/k, \cdots, 1/k)$ for some integer $k > 1$ and $\lambda = k$. Lind pointed out that this cannot happen. Otherwise, let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of the matrix representing T. Then the λ_i are algebraic integers and hence so is $1/k = \pm \pi_{|\lambda_i|<1} \lambda_i$. But a rational algebraic integer is an integer and $k > 1$.

Now let $T: M \to M$ be a general C^2 Anosov diffeomorphism preserving the smooth measure m. For $\mathcal{B}$ a Markov partition $\lim F_{\mathcal{B},n,T}$ will exist, again being normal or point mass [5]. One would have $\mathcal{A} \sim_{\text{bd.}} \mathcal{B}$ for $\mathcal{A}$ a small smooth partition *if* one could get $m(\partial_\varepsilon \mathcal{B}) \leqq c\varepsilon$. Then $\lim_n F_{\mathcal{A},n,T}$ would equal $\lim_n F_{\mathcal{B},n,T}$. I suspect this in fact is the case and that $\lim_n F_{\mathcal{A},n,T}$ is a point mass iff the entropy $h_m(T)$ equals the topological entropy of T.

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